PERMUTATIONS & COMBINATIONS

Fundamental Principle of Counting:

There are two basic principles of counting:

Principle of addition

Suppose a work A can be performed in m ways and B can be performed in n ways and both cannot occur simultaneously, then the number of ways of performing one of them is (m + n).

This rule is also applicable for two or more exclusive events.

Principle of multiplication

Suppose a work A can be performed in m ways and B can be performed in n ways. Also A and B are not related in the sense that B can be performed in n ways regardless of the outcome of A. Then both A and B can be performed in m $\times$ n ways.

If

	Event A	$\xrightarrow[\text{can occur in}]{}$	m ways.	
	Event B	$\xrightarrow[\text{can occur in}]{}$	n ways.	
Then,	At least one (A **or** B)	$\xrightarrow[\text{can occur in}]{}$	m + n ways	(Sum Rule)
	A **and** B (both simultaneously)	$\xrightarrow[\text{can occur in}]{}$	m $\times$ n ways	(Product Rule)

> Always keep in mind that whenever there is '**OR**' then **add**, and whenever there is '**AND**', then **multiply**.

Examples:

1. At an ice-cream parlour four different ice-cream flavours are available; chocolate, vanilla, strawberry and mango. Each ice-cream is available in either a cone or a cup. How many choices are available to buy an ice-cream?
2. In a readymade store, 4 brands of shirts and 3 brands of trousers are available.
   Each brand offers three different colours.
   A) How man choices are available for Sam to buy a shirt or a trouser?
   B) If Sam wants to buy a shirt and a trouser, how man choices are available for him?

DEFINITIONS

Permutations: Each of the different arrangements, obtained by taking some, or all, of a number of things, is called a ***Permutation***.

It is denoted by $^nP_r = \frac{n!}{(n-r)!}$ where $1 \le r \le n$.

Combinations: Each of the different groups or collections or selections that can be made by taking some, or all, of a number of things, irrespective of the order in which the things appear in the group, is called a ***Combination***.

It is denoted by $^nC_r = \frac{n!}{(n-r)!\,r!}$ where $1 \le r \le n$

Example: Suppose, there are four quantities A, B, C, D. The different orders of arrangements of these four quantities by taking three at a time are $^4P_3 = 4\,! = 24$ ways and they are

ABC, ACB, ABD, ADB, ACD, ADC,
BAC, BCA, BAD, BDA, BCD, BDC
CAB, CBA, CDB, CBD, CDA, CAD
DAB, DBA, DAC, DCA, DBC, DCB

Thus, each of the 24 arrangements, of the four quantities A, B, C, D by taking three at a time, are each called a ***permutation***. Hence, it is clear that the number of permutations of four things taken three at a time is 24.
Now, if you want to choose or select three quantities at a time among these four then total number of ways are $^4C_3$ = 4 or exactly are ABC, ABD, ACD, BCD.

PERMUTATIONS AND COMBINATIONS

Most important here is how to distinguish between permutation & combination i.e. where to apply permutation & where combination.

Permutations
↓
For **arranging** the things like sitting arrangement, alphabets arrangement (making words), forming numbers etc.
AND
Combination
↓
For **selection/grouping** like making a team

Remember: – In permutation, order of arrangement is taken in account, whereas in combination order is immaterial.

NOTE:
1. $n\,! = n(n-1)\,(n-2)\,(n-3)\ldots..1$
2. $0\,! = 1$ (by definition)

Permutations: Summary of different basic results

- $^{n}P_r = \frac{n!}{(n-r)!}$ i.e. Permutations of n different things taken 'r' at a time
- $^{n}P_r = n(n-1)(n-2)\ldots.r$ [e.g. $^{7}P_4 = 7.6.5.4$]

- **Circular permutation**
  Number of circular permutations of n things taken all at a time = **(n – 1)!**
  Number of circular permutations of n different things taken r at a time = $\frac{^{n}P_r}{r}$.
- **Special case:** When the clockwise and anti-clockwise arrangements are the same, as in case of garland, necklace etc., then –
  - the total number of circular permutations of n things taken all at a time is $\frac{(n-1)!}{2}$.
  - the total number when taken r at a time all will be $\frac{^{n}P_r}{2r}$.
- **The number of permutations when things are not all different:** If there be n things, p of them of one kind, q of another kind, r of still another kind and so on, then the total number of permutations = $\frac{n!}{p!q!r!}$.
- **Permutation with repetitions:** The number of permutations of n different things taking r at a time when each thing may be repeated any number of times in any permutation is given by
  $(n \times n \times n \times n \times n \ldots\ldots\ldots\ldots\ldots r \text{ times})$ i.e. $\mathbf{n^r}$ ways.
- The total number of arrangements of n things taken r at a time, in which a particular thing always occurs is = $r\ ^{n-1}P_{r-1}$.
- The total number of permutations of n different things taken r at a time in which a particular thing never occurs is = $^{n-1}P_r$.

Combinations: Summary of different basic results

Number of combinations of n dissimilar things taken 'r' at a time is denoted by $^{n}C_r$ & is given

by $^{n}C_r = \frac{n!}{(n-r)!r!}$.

- We have $^{n}C_r = \frac{n!}{(n-r)!r!} \Rightarrow {}^{n}C_r = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots.3.2.1}{\{(n-r)(n-r-1)\ldots3.2.1\}\{1.2.3\ldots.r\}}$

  $\Rightarrow {}^{n}C_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots.r}$.

  Sometimes this form of $^{n}C_r$ is also very convenient to use.

  For example $^{10}C_3 = \frac{10.9.8}{1.2.3} = 120$.
- $^{n}C_n = {}^{n}C_0 = 1$.

  Proof: – We have $^{n}C_r = \frac{n!}{(n-r)!r!}$. Putting r = n, we obtain $^{n}C_r = \frac{n!}{(n-n)!n!} = \frac{n!}{n!0!} = 1$ [Because 0! = 1]

  Putting r = 0, we obtain $^{n}C_0 = \frac{n!}{(n-0)!0!} = \frac{n!}{n!} = 1$.

  Thus $^{n}C_n = {}^{n}C_0 = 1$.

- $\mathbf{{}^nC_r = {}^nC_{n-r}}$ for $0 \le r \le n$

  Proof: **–** We have, ${}^nC_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r$

  Remark:

  The use of this property simplifies the calculation of ${}^nC_r$ when r is large.

For example**,** If we want to calculate ${}^{20}C_{19} = {}^{20}C_{20-19} = {}^{20}C_1 = 20$.

- ${}^nC_r = \dfrac{n!}{(n-r)!r!} = \dfrac{1}{r!}\left(\dfrac{n!}{(n-r)!}\right) = \dfrac{{}^nP_r}{r!}$
- $\mathbf{{}^nC_x = {}^nC_y \Rightarrow x = y \text{ or } x + y = n.}$
- Let n and r be non-negative integers such that $r \le n$. Then, $\mathbf{{}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r}$
- Let n and r be non-negative integers such that $1 \le r \le n$. Then. $\mathbf{{}^nC_r = \frac{n}{r}.{}^{n-1}C_{r-1}}$
- If $1 \le r \le n$, then $\mathbf{n.\ {}^{n-1}C_{r-1} = (n - r + 1)\ {}^nC_{r-1}}$
- Number of combinations of n different things taken 'r' at a time in which p particular things will always occur is given by $\mathbf{{}^{n-p}C_{r-p}}$.
- No. of combinations of n dissimilar things taken 'r' at a time in which 'p' particular things will never occur is $\mathbf{{}^{n-p}C_r}$.
- If n is even, then the greatest value of ${}^nC_r$ $(0 \le r \le n)$ is $\mathbf{{}^nC_{n/2}}$
- If n is odd, then the greatest value of ${}^nC_r$ $(0 \le r \le n)$ is ${}^nC_{\frac{n+1}{2}}$ or ${}^nC_{\frac{n-1}{2}}$
- The number of ways in which (m + n) things can be divided into two groups containing m & n things respectively $= {}^{(m+n)}C_n = \dfrac{(m+n)!}{m!n!} = {}^{(m+n)}C_m$
- If 2m things are to be divided into two groups, each containing m things, then the number of ways is given by $\dfrac{(2m)!}{2(m!)^2}$
- The number of ways to divide n things into different groups, one containing p things, another containing q things and so on $= \dfrac{(p+q+r.....)!}{p!q!r!.....}$ where $\{n = p + q + r + \ldots\}$
- Total number of combinations of n things, taking some or all at a time, when p of them are alike of one kind, q of them are alike of another kind & so on is given by $\{(p + 1)(q + 1)(r + 1) \ldots\} - 1$ where $[n = p + q + r + ..]$
- The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.
  (In making selections each item can be dealt with in two ways; it is either selected or rejected and corresponding to each way of dealing with one item, any one of the other items can also be dealt with in 2 ways. So, the total number of ways of dealing with n items is 2^n. But these 2^n ways also include the case when all the items are rejected. Hence, required number of ways $= 2^n - 1$.)

Ex. A man has 6 friends; in how many ways may he invite one or more of them to dinner?

Sol. A man has to select some or all of his friends. So, required number of ways $= 2^6 - 1 = 63$.

SOME IMPORTANT RESULTS:

- The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the groups is *not important*, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}$$

The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of the groups is *important*, is

$$\left(\frac{(mn)!}{(n!)^m}\right)\frac{1}{m!}\times m! = \frac{(mn)!}{(n!)^m}$$

Ex. In how many ways can a pack of 52 cards be divided equally among four players in order?

Sol. Here 52 cards are to be divided into four equal groups and the order of the groups is important.
So, required number of ways is

$$\left(\frac{52!}{(13!)^4 4!}\right)4! = \frac{52!}{(13!)^4}$$

- The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0, 1, 2, or more items ($\leq$ n) is ${}^{n+r-1}C_{r-1}$

OR

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

Ex. The total number of ways in which 30 mangoes can be distributed among 5 persons

Sol. Required number is ${}^{30+5-1}C_{5-1} = {}^{34}C_4$.

- The total number of ways of dividing n identical items among r persons, each one of whom, receives at least one item is ${}^{n-1}C_{r-1}$

OR

The number of ways in which n identical items can be divided into r groups such that blank groups are not allowed, is ${}^{n-1}C_{r-1}$.

Ex. Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all balls.

Sol. The required number of ways is the number of ways of distributing 5 items among 3 persons so that a person receives at least one item = ${}^{5-1}C_{3-1} = {}^4C_2 = 6$.

- If n items are arranged in a row, then the number of ways in which they can be rearranged so that no one of them occupies the place assigned to it, is

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right]$$

Ex. There are 6 letters and 6 directed envelopes. Find the number of ways in which all letters are put in the wrong envelopes.

Sol. The required number of ways = $6!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right] = \frac{6!}{2!}-\frac{6!}{3!}+\frac{6!}{4!}-\frac{6!}{5!}+\frac{6!}{6!}$

= 360 – 120 + 30 – 6 + 1 = 265

Always remember:

(i) Number of lines with n points = $^nC_2$.

For making a line exactly two points are required. So number of ways in which we can choose two points out of n point is $^nC_2$.

Combination is used here because line from A to B in same as B to A. So AB & BA are same.

(ii) n lines can intersect at a maximum of $^nC_2$ points.

(iii) Number of triangles with n points = $^nC_3$.

(iv) Number of diagonals in n sided polygon = $^nC_2 - n$

EXAMPLES

1. How many different words can be formed out of the letters of the word 'DRAUGHT' so that vowels are not separated?

Sol. Total 7 letters are there, the two vowels are not to be separated $\Rightarrow$ treat them to be one $\Rightarrow$ number of words when vowels are considered to be one = 6! But 'a' and 'u' can themselves be arranged in two ways. $\Rightarrow$ Different words are =2 (6!) = 1440

2. How many different words can be formed with the letters of word 'ORDINATE'
(1) So that the vowels occupy odd places?
(2) Beginning with 'O'
(3) Beginning with 'O' and ending with 'E'

Sol. (1) ORDINATE contains 8 letters: 1, 2, 3, 4, 5, 6, 7, 8. It has 4 odd places, 4 vowels.

$\Rightarrow$ number of arrangements of the vowels 4 ! Also number of arranging consonants is 4!

$\Rightarrow$ Number of words = $4! \times 4! = (4 \times 3 \times 2 \times 1)^2 = 576$.

(2) When O is fixed we have only seven letters at our disposal $\Rightarrow$ Number of words = 7! = 5040

(3) When we have only six letters at our disposal, leaving 'O' and 'E' which are fixed.

Number of permutations = 6! = 720

3. How many different words can be formed with the letters of the word 'TRIANGLE'?

Sol. 8 letters $\Rightarrow$ Number of permutations = 8! = 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 40320

4. How many different words beginning and ending with a consonant, can be made out of the letters of the word 'EQUATION'?

Sol. There are total 8 letters, 3 consonants and 5 vowels.

So arrangement of 1st and last places can be made in $^3P_2$ ways.

Also the remaining 6 letters can be arranged in 6! Ways.

$\Rightarrow$ the required number of words = $^3P_2 \times 6! = 3 \times 2 \times 6! = 4320$.

5. If $^{18}C_r = {}^{18}C_{r+2}$ find $^{10}C_r$ and $^rC_5$.

Sol. Since $^nC_r = {}^nC_{n-r} \Rightarrow {}^{18}C_{18-r} = {}^{18}C_{r+2}$ $\quad 18 - r = r + 2, \Rightarrow r = 8$

$\Rightarrow$ $^{10}C_r = {}^{10}C_8 = 10! / (8! \times 2!) = 10 \times 9/2! = 45$;

And $^rC_5 = {}^8C_5 = 8! / (5! \times 3!) = 8 \times 7 \times 6/3 \times 2 \times 1 = 56$.

6. If $^nC_7 = {}^nC_9$ find n.

Sol. $^nC_7 = {}^nC_{n-7} = {}^nC_9 \Rightarrow n - 7 = 9$ or $n = 7 + 9 = 16$.

7. If $^nC_{10} = {^nC_{15}}$, find $^{27}C_n$.

Sol. Since $^nC_r = {^nC_{n-r}} \Rightarrow {^nC_{10}} = {^nC_{n-10}} = {^nC_{15}}$

$\Rightarrow$ n – 10 = 15 or n = 10 + 15 = 25

$\Rightarrow {^{27}C_{25}} = 27!/(25!\ 2!) = 27 \times 26/(1 \times 2) = 351$

8. If $^{2n+1}P_{n-1} : {^{2n-1}P_n} = 3 : 5$, find n.

Sol. Here $^{2n+1}P_{n-1}\ /\ {^{2n-1}P_n} = 3/5$

or $\dfrac{\dfrac{(2n+1)!}{[(2n+1)-(n-1)]!}}{\dfrac{(2n-1)!}{[(2n-1)-n]!}} = 3/5$ or $\dfrac{(2n+1)!}{(n+2)!} \times \dfrac{(n-1)!}{(2n-1)!} = 3/5$

or $\dfrac{(2n+1).2}{(n+2)(n+1)} = 3/5 \Rightarrow 20n + 10 = 3n^2 + 9n + 6$

$\Rightarrow 3n^2 - 11n - 4 = 0 \Rightarrow 3n^2 - 12n + n - 4 = 0 \Rightarrow (n-4)(3n+1) = 0$

$\Rightarrow$ n = 4 the other value being inadmissible.

9. If $^nP_r = {^nP_{r+1}}$ and $^nC_r = {^nC_{r-1}}$, find n and r.

Sol. $^nP_r = n!/(n-r)!$ and $^nP_{r+1} = n!/\{n - (r+1)\}!$

$\Rightarrow n!/(n-r)! = n!/(n-r-1)! \Rightarrow (n-r-1)! = (n-r)! = (n-r)(n-r-1)!$

$n - r - 1 = 0$ …..(1)

Also $^nC_r = n!/r!(n-r)!$ and $^nC_r = n!/(r-1)!(n-r+1)!$

$\Rightarrow n!/r!(n-r)! = n!/(r-1)!(n-r+1)! \Rightarrow r!(n-r)! = (r-1)!(n-r+1)!$

But n – r = 1 by ….(1) $\Rightarrow r!\,1! = (r-1)!2! \Rightarrow$ r = 2 and so n = 3

10. In how many ways can a party of 5 persons be formed from 6 men and 5 ladies consisting of 3 men and two ladies?

Sol. Number of ways of selecting men = $^6C_3 = 6!/3!.3! = (6 \times 5 \times 4)/(3 \times 2 \times 1) = 20$

Number of ways of selecting ladies = $^5C_2 = 5!/2!.3! = (5 \times 4)/(2 \times 1) = 10$

$\Rightarrow$ Number of parties = 20 x 10 = 200

11. A party of four persons is to be selected from 8 Hindus and 3 Muslim so as to include at least one Muslim. Find the possible number of ways?

Sol. (1) Suppose we take only one Muslim. No of selections of Muslims = $^3C_1 = 3$

Also number of selections of Hindus = $^8C_3 = 8!/3!.5! = (8 \times 7 \times 6)/(3 \times 2 \times 1) = 56$

$\Rightarrow$ Number of parties = 56 x 3 = 168

(2) When we select 2 Muslim and 2 Hindus, Number of ways of selecting Muslims = $^3C_2 = 3$

Number of ways of selecting Hindus $^8C_2 = 8 \times 7/2 = 28$

$\Rightarrow$ Number of parties = 28 x 3 = 84

(3) When to take 3 Muslim and one Hindus,

$\Rightarrow$ Number of ways of selecting Muslims = $^3C_3 = 1$

$\Rightarrow$ Number ways of selecting Hindus = $^8C_1 = 8$

$\Rightarrow$ Number of parties = 8

$\Rightarrow$ Final answer = Total number of parties = 168 + 84 + 8 = 260.

Alternative method:

Possible number of ways = Number of ways of selecting 4 persons out of 11 persons (8 + 3) – Number of ways of selecting 4 persons out of 8 Hindus.

12. How many diagonals are there in an octagon?

Sol. The diagonal is obtained by joining two angular points which are 8 in number. $\Rightarrow$ Number of the lines joining the two points = $^8C_2$ = 8! /2!.6! = 28. But it includes 8 sides also as they too are obtained by joining two points $\Rightarrow$ Number of diagonals = 28 – 8 = 20.

13. How many triangles can be formed from a hexagon?

Sol. 3 non-collinear points form a triangle
$\Rightarrow$ Number of triangles = $^6C_3$ = (6 x 5 x 4) / (3 x 2 x 1) = 20.

14. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?

Sol. We are to select 3 consonants out of 7 and 2 vowels out of 4. 3 consonants can be selected out of 7 in $^7C_3$ ways, and 2 vowels can be selected out of 4 in $^4C_2$ ways. Take any one of these relations. There are 5 letters in it and they can be arranged in 5! ways. $\Rightarrow$ For $^7C_3$ x $^4C_2$ combinations, the number of arrangements = $^7C_3$ x $^4C_2$ x 5! = 2200.

15. How many different words can be formed of the letters of 'INTERMEDIATE' all taken together?

Sol. 12 letters : 2 'i' ; 1 'n' ; 2 't' ; 3 'e' ; 1 'r' ;1 'm' ; 1 'd' ; 1 'a' . Thus we see that out of 12 letters 2 are of one kind, 2 of another and 3 of another $\Rightarrow$ No. of arrangements = 12!/(3! x 2! x 2!)

$$= \frac{12x11x10x9x8x7x6x5x4x3x2x1}{3x2x1x2x1x2x1} = 19958400$$

16. How many different words can be formed from the letters of the word 'SERIES' taken all together?

Sol. SERIES contains 6 letters; 2 'S' ; 2 'E' ; 1 'I' ; 1 'R'
$\Rightarrow$ Number of words = 6! /2!.2! = (6 x 5 x 4 x 3 x 2 x 1) / (2 x 1 x 2 x 1) = 180.

17. How many words can be formed from the letters of the word 'INDIA' taken all together?

Sol. There are only 5 letters of which two are I's. Therefore required no. of words = 5!/2! = 60

18. How many numbers, each consisting of four different digits, can be formed with the digits 0, 1, 2, 3?

Sol. With 0, 1, 2, 3 we can form 4! = 24 numbers. But it includes the numbers in which 0 occurs in thousand's place and they will be numbers only of 3 digits. Such numbers will be 3! = 6.
$\Rightarrow$ (24 – 6) = 18 numbers can be formed which will be of four digits.

19. How many numbers lying between 3000 and 4000 and divisible by 5 can be made with the digits 3, 4, 5, 6, 7 and 8? (Digits are not to be repeated in any number)

Sol. Every number between 3000 and 4000, which is divisible by 5 and which can be formed by the given digits, must contain 5 in unit's place and 3 in thousand's place Thus we are left with four digits out of which we are to place two between 3 and 5, which can be done in $^4P_2$ = 12 ways. Hence 12 numbers can be formed.

20. In how many ways can 18 different books be divided equally among three students? Also find in how many ways three piles can be formed?

Sol. Since each will get six books, the books can be divided in $18!/(6!)^3$ ways.
Again piles can be made in $\{18!/3!.(6!)^3\}$ ways.